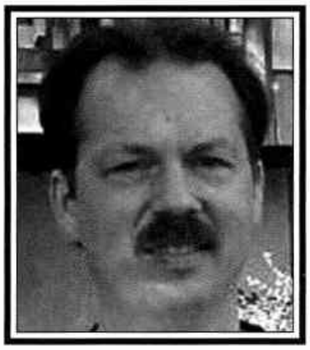

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PLACE-VALUE AND THE ELUSIVE DECIMAL POINT

"When you multiply a number by 10, you move the decimal point one step to the right. When you divide by 10 you move the decimal point one step to the left."

The above "rule" is taught in many upper elementary and middle school classrooms and is found in many textbooks in use across New York State and the country. The problem is that, while this rule may work for getting a correct answer, it is mathematically incorrect. We resort to even more mystifying rules when we multiply and divide with decimal numbers.

In reform texts, estimation is the preferred method for determining where the decimal point should show up (see Van de Walle, 2001, p.256-257). For example, 2.3 x 1.7 is approximately 2 x 2, which results in a product close to four. First children would ignore the decimal points and then later place it according to their estimate. This method relies on children gaining insight into the fact that operations with numbers that have a decimal point in them are identical to those having numbers without a decimal point.

In traditional texts students find the number of digits after the decimal point in both factors of a multiplication problem and then multiply without the decimal points present. After calculating the product they then place the decimal point in this product by counting the total digits from the right to the left. This works, if all we care about is a correct answer. This article focuses on why this method works and why it is not a wise choice. An alternative method is proposed that is mathematically sound and makes use of the algebraic properties and patterns of the place-value system.

The Place-value System

What really happens to a number when multiplied or divided by a power of 10 requires a deeper understanding of the decimal place-value system. We focus on numerical and geometric relationships and patterns in this system. These are indicated in figure 1. In your class you can develop this figure over time. It is an extension of the place-value mats used in many elementary classrooms. By using place-value materials the geometric pattern of cube, long, flat (reading from right to left) or cube, flat, long (reading from left to right) helps children concretely see that each piece to the left of any given piece is ten times higher in value and each piece to the right of any given piece is ten times lower in value. It is important that children learn that this is *independent* of where you are in the pattern. Another important idea is that the pattern goes in both directions, no matter where you start. We will get back to this in a moment. Numerically you can develop several patterns. One pattern is that no matter where you are in the system a multiplier of 10 causes a trading up to the left and a divisor of 10 causes a trading down to the right. A second pattern is that every three places constitutes a multiplier of 1000. If we start from the ones place to the left, this causes a name change every three places. In the United States we represent this pattern in numbers by using a comma. This numerical pattern is also visible geometrically in the three-dimensional model of cube, long (ten cubes), and flat (ten longs), cube (ten flats), and so on.

The Decimal Point

Since the above mentioned patterns can start anywhere, we need a system to communicate to others where we started our pattern. We agree to always start our place-value system relative to the *ones* place and the decimal point tells us where it is located. This is the only role that we have for the decimal point. By virtue of this role it is impossible for a decimal point to move anywhere. It must indicate the ones place and therefore must be located next to it. The ones place has a central role in the place-value system. Note that in any given number, such as 123.321, the decimal point is not the center, but rather the ones place is (see figure 2).

GIANT CUBE	BIG FLAT	BIG LONG	BIG CUBE	FLAT	LONG	CUBE	SMALL FLAT	SMALL LONG	SMALL CUBE
MILLIONS	HUNDRED THOUSANDS	TEN THOUSANDS	THOUSANDS	HUNDREDS	TENS	ONES	TENTHS	HUNDREDTHS	THOUSANDTHS
1,000,000	100,000	10,000	1,000	100	10	1	0.1	0.01	0.001
10^6	10^5	10^4	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}
				x10 ÷10					
					x1000 ÷1000				
						PERCENT HUNDRED	*PERCENT TENS*	*PERCENT ONES*	*PERCENT TENTHS*

Figure 1. Numerical and Geometric Patterns in the Place-Value System

Hundreds	Tens	ONES		Tenths	Hundredths	Thousandths
1	2	3	.	1	2	3

Figure 2. Symmetry in Place-Value

Take another look at figure 1 to see that names for place-values are symmetrical around the ones place. At the bottom of this figure you can see the place-value system for percents. Note that the choice for the ones place has changed by two places. This means then that the calculations for percents are identical to those in the regular base-ten system, with the difference that we have chosen the ones place to be in the hundredths place of our standard system. This knowledge ties much of what seems separate together. It may be of interest to students to see that how we identify the starting point (ones place) has changed over time. You can see some examples of this in figure 3.

O O |||	8th Century AD, China. *O* was the symbol for both 0 and the decimal point. On the left is .03
3 | 25	1530: Christoff Rudolf in his book *Exemel Buchlin* represented 3.25 as on the left.
3, 2' 5'' 7'''	1558: Simon Stevin recommended that governments adopt the decimal system. He wrote 3.257 as on the left
3 •275	1619: John Napier used a raised point in his work, as was common at the time in England. 3.275 is indicated on the left
3 275	17th Century, Henry Briggs identified the fractional parts by underlining.
3 | 275	Around that time you can also find a combination of a bar and underlining
3 •275 3,275 3.275	In the nineteenth century still a variety of notations were used. England France and Germany United States

Figure 3 History of the Decimal Symbol.

Operations with Decimals

In adding and subtracting with decimal numbers, the decimal points help us line up the digits of each number in the proper place-value. Here the decimal point stays in one spot. In multiplication and division, the same must be true, because the decimal point never moves. What does happen when we multiply or divide a number with a power of 10? Let us look at an example. We can look at what happens in 17×10 and next at $130 \div 10$.

For 17×10, we often teach just to "add a zero." However we need to consider what happens to each digit. If one *ten* is multiplied by 10, we get 10 *tens*. Earlier children have learned that these (10 longs) can be traded up for one *hundred* (flat). Effectively that makes the digit 1, move from the tens place to the hundreds place. The same thing happens to the digit 7, it moves from the ones place to the tens place. In order for us to communicate this we must write a 0 in the ones place, because that is the start point of our place-value system. What we can learn from this is that what moves are the digits and not the decimal point. We understood this by trading up in the case of multiplication. In the case of division you can see that each digit moves one place down upon division by ten. In this case we must trade down to accomplish the division: the one hundred (flat) trades down to 10 tens (longs) and the three tens (longs) trade down to 30 ones (cubes).

The Digits Move!

Whether or not numbers have a decimal point in them the operations must work in exactly the same way. In figure 4, you can see an example for and . As we can see, by aligning the decimal points, and therefore the place-values, the digits move up and down two places respectively.

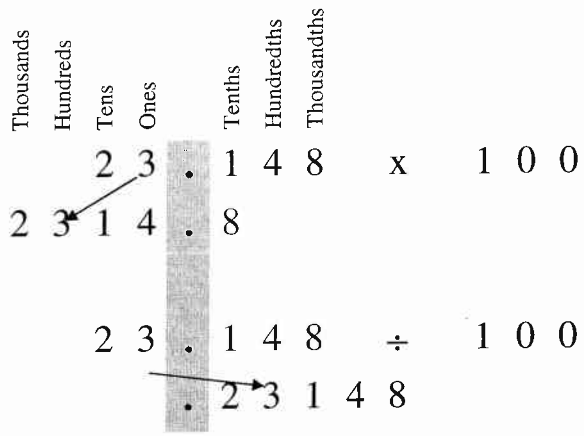

Figure 4 The Digits Move!.

We finally need to look at what happens when we multiply two numbers that both have a decimal point in them. For the scope of this article, division of two decimal numbers is referred to the traditional method of writing the quotient in an equivalent form without decimal points. Even here the digits move and not the decimal point. However, the emphasis should be on equivalence, rather than "ignoring" the decimal points. It is proposed here to consistently use the area model or partial product model for multiplication. This model

more clearly uses the distributive property. The standard algorithm is simply a shorter version of this principle. This method also models the algebraic way of multiplying (often presented with the acronym FOIL), which is a major additional benefit. In figures 5a and 5b, you see compared to , represented with the area model and in traditional numerical column form.

	20	3
10	10x20 = 200	10x3 = 30
7	7x20 = 140	7x3 = 21

	2	0.3
1	1x2 = 2	1x.0.3 = 0.3
0.7	0.7x2 = 1.4	0.7x0.3 = 0.21

Figure 5a. The area model for Multiplication

2 . 3	2 3
1 . 7 x	x 1 7
. 2 1	2 1
1 . 4 0	1 4 0
. 3 0	3 0
2 . 0 0	2 0 0
3 . 9 1	3 9 1

Figure 5b. The decimal point does not move in Partial Products

Note that in the decimal points all align, and therefore all the place-values align. The multiplication sign has been switched to the right side in order to keep the student focused on proper place-value alignment. It must also be noted that students need to develop a good sense of what happens when you multiply 0.7 with 0.3. It does not make sense to go this route when students do not yet have a good grasp on this. In that case the estimation method may benefit students to know how the place-values should align.

Concluding Remarks

In this article we have seen that in all circumstances in which we deal with decimal numbers, the decimal point never moves. Explanations of what happens when numbers are multiplied and divided are rooted in the place-value system. In this system the role of the decimal point is to tell us where the system starts, i.e. where the ones place is. This is not a trivial matter. Students need to understand this well so that they learn the underlying structures. This, in turn, prepares elementary and middle school students for algebra.

References

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Van de Walle, J.A. (2001). *Elementary and middle school mathematics: Teaching developmentally, Fourth Edition.* New York, NY: Addison Wesley Longman, Inc.